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STEADY-STATE OSCILLATIONS OF AN ELLIPTIC PLATE

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An asymptotic method is given for constructing a stress-strain state of an elliptic plate subjected to dynamic loading, using a three-dimensional formulation. Investigations are carried out for the case of skew-symmetric (bending) oscillations of the plate relative to the middle surface, although the method can also be used in the case of symmetric oscillations of the plate.

Let us consider forced steady-state oscillations of an isotropic, homogeneous elliptic plate of thickness $2h$, with semiaxes a and b (Fig. 1). We assume that the plane edges of the plate are stress-free and the loading acts only on the cylindrical side surface. We seek the displacement vector in the form $\mathbf{a}_t = \mathbf{a}\ (x, y, z)e^{i\omega t}$, where ω denotes the frequency of the forced vibrations. Satisfying the Lamé system of differential equations and the boundary conditions at the plane edges and using the Lur'e [1] symbolic method, we find that the stress-strain state of the plate consists of two states. One of these states id described by the vector of the amplitude displacement functions

$$u_n^{(1)} = R\sum_{m=1}^{\infty} \Lambda_m \frac{\partial \Psi_m}{\partial n}, \quad u_s^{(1)} = R\sum_{m=1}^{\infty} H^{-1}\Lambda_m \frac{\partial \Psi_m}{\partial s} \tag{1}$$

$$w^{(1)} = R\sum_{m=1}^{\infty} [2\mu_m \cos\lambda\alpha_m\zeta \cos\lambda\beta_m - (\mu_m + \alpha_m^2)\cos\lambda\beta_m\zeta \cos\lambda\alpha_m]\,\Psi_m$$

$$\Lambda_m = 2\alpha_m \sin\lambda\alpha_m\zeta \cos\lambda\beta_m - (\mu_m + \alpha_m^2)\cos\lambda\alpha_m \frac{\sin\lambda\beta_m\zeta}{\beta_m}$$

$$\lambda = h/R, \quad \zeta = z/h, \quad \alpha_m^2 = \mu_m + \Omega^2, \quad \beta_m^2 = \mu_m + (1-2\nu)\,\Omega^2/[2(1-\nu)]$$

$$\Omega^2 = \omega^2 R^2 \rho_1 G^{-1}, \quad H = 1 + nR\rho^{-1}$$

$$\Delta\Psi_m = \mu_m \Psi_m \tag{2}$$

$$(\mu_m + \alpha_m^2)^2 \cos\lambda\alpha_m \frac{\sin\lambda\beta_m}{\beta_m} - 4\mu_m\alpha_m \sin\lambda\alpha_m \cos\lambda\beta_m = 0 \quad (m = 1, 2, \ldots) \tag{3}$$

where s, n denotes a local system of dimensionless, orthogonal coordinates attached to the contour Γ of the plate in the x, y -plane (Fig. 1), ρ is the radius of curvature of the contour Γ, R is the characteristic dimension of the plate in the x, y - plane (the smallest radius of curvature of the ellipse), G is the shear modulus, ν

is the Poisson's ratio , ρ_1 is the material density of the plate and μ_m are the roots of the Rayleigh-Lamb equation (3).

The other stress-strain state of the plate is described by the vector of the amplitude displacement functions

$$u_n^{(2)} = R\sum_{k=1}^{\infty}\frac{\sin\lambda\chi_k}{\chi_k}\frac{1}{H}\frac{\partial B_k}{\partial s}, \quad u_s^{(2)} = -R\sum_{k=1}^{\infty}\frac{\sin\lambda\chi_k}{\chi_k}\frac{\partial B_k}{\partial n} \tag{4}$$

$$w^{(2)} = 0, \quad \Delta B_k - \sigma_k{}^* B_k = 0$$
$$\cos\lambda\chi_k = 0, \quad \chi_k{}^2 = \sigma_k{}^* + \Omega^2 \quad (k = 1, 2, \ldots) \tag{5}$$

In the case of bending oscillations, the Lamb equation (3) has two real roots (of order $1 / \lambda$) within the range of variation of Ω $(\Omega\lambda < 4.7)$, and an enumerable set of complex roots (of order $1 / \lambda^2$) the asymptotic expansions of which are given in [2].

Let us denote the complex roots of the Lamb equation by $\gamma_p{}^2 / \lambda^2$ and the corresponding solutions of (2) by $C_p\ (n, s),\ p = 1, 2, \ldots$. Paper [3] gives asymptotic representations of these solutions in terms of the values of the functions C_p at the boundary Γ. The solution in question is a boundary layer solution. Below we shall utilize the following asymptotic expansions of the normal derivatives of the functions C_p at the contour Γ :

$$\left.\frac{\partial C_p}{\partial n}\right|_\Gamma = \frac{1}{\lambda}\sum_{i=0}^{\infty}\lambda^i S_{pi} c_p(s), \quad \left.\frac{\partial^2 C_p}{\partial n^2}\right|_\Gamma = \frac{1}{\lambda^2}\sum_{i=0}^{\infty} Q_{pi} c_p(s)\lambda^i \tag{6}$$
$$c_p(s) = C_p(n, s)\,|_\Gamma$$

where the operators S_{pi} and Q_{pi} are given below.

Denoting the roots of (5) by $\sigma_k^* = \sigma_k^2 / \lambda^2$, we obtain

$$\sigma_k{}^2 = \frac{\pi^2(2k-1)^2}{4} - \lambda^2\Omega^2 \quad (k = 1, 2, \ldots)$$

The expressions (6) hold for $B_k\ (s, n)$ when $\lambda\Omega < \pi / 2$, provided that C_p is replaced by B_k, c_p by b_k and γ_p by σ_k. When $\lambda\Omega \geqslant \pi / 2$, then the quantity σ_1 becomes purely imaginary in the range of variation of Ω in question and the solution corresponding to σ_1 will no longer be a boundary layer solution [4]. In this case we construct the solution using the Mathieu functions in the manner analogous to that used below for the real roots of the Lamb equation.

We write the solutions of the Helmholtz equation corresponding to the real roots of the Lamb equation (we shall call these solutions, in what follows, the penetrating solutions) in the elliptical ξ, η -coordinates, in terms of the Mathieu functions [5]. The dimensionless Cartesian and elliptical coordinates are connected by the formulas (ε is the eccentricity of the ellipse and $\xi = \xi_0$ corresponds to the plate contour Γ)

$$x_1 = \frac{x}{R} = \frac{\varepsilon^2}{1-\varepsilon^2}\operatorname{ch}\xi\cos\eta, \quad y_1 = \frac{y}{R} = \frac{\varepsilon^2}{1-\varepsilon^2}\operatorname{sh}\xi\sin\eta$$
$$R = a(1-\varepsilon^2), \quad 0 \leqslant \eta \leqslant \pi, \quad 0 \leqslant \xi \leqslant \xi_0$$

Solution of (2) written in the elliptical coordinates with the physical meaning of the problem taken into account, has the form [5]

$$\Psi_i = \sum_{m=0}^{\infty} A_m^{(i)} \operatorname{Ce}_m (q_i, \xi) \operatorname{ce}_m (q_i, \eta) + \sum_{m=0}^{\infty} B_m^{(i)} \operatorname{Se}_m (q_i, \xi) \operatorname{se}_m (q_i, \eta)$$

$$q_i = \mu_i \varepsilon^2 [4(1-\varepsilon^2)]^{-1} \quad (i = 1, 2)$$

We shall assume that the following system of forces (Fig. 1) is given at the cylindrical part of the plate surface:

$$\sigma_n = 2GN(\zeta, s) e^{i\omega t}, \quad \tau_{ns} = 2GT(\zeta, s) e^{i\omega t},$$

$$\tau_{nz} = 2GZ(\zeta, s) e^{i\omega t}$$

To simplify the arguments, we shall consider the oscillations symmetrical about the ellipse axes, i.e. $B_m^{(i)} = 0$; $i = 1, 2$; $m = 0, 1, 2, \ldots$

We construct the stress-strain state in the plate as a sum of three states: the penetrating state, the potential and the vortical state. Then we have the following relations at the side surface of the plate:

$$N = \sum_{i=1}^{2} \sum_{m=0}^{\infty} A_m^{(i)} \Big\{ J_0^{-2} \Lambda_i \Big[\frac{\partial^2}{\partial \xi^2} - \frac{1}{\operatorname{ch} 2\xi_0 - \cos 2\eta} \Big(\operatorname{sh} 2\xi_0 \frac{\partial}{\partial \xi} - \sin 2\eta \frac{\partial}{\partial \eta} \Big) \Big] + \frac{\Omega^2 \nu}{2(1-\nu)} (\mu_i + \alpha_i^2) \frac{\sin \lambda \beta_i \zeta}{\beta_i} \cos \lambda \alpha_i \Big\} \times$$
$$\operatorname{Ce}_{2m}(q_i, \xi)\big|_{\xi=\xi_0} \operatorname{ce}_{2m}(q_i, \eta) + \lambda \sum_{p=1}^{\infty} Q_p(\zeta) \frac{\partial^2 C_p}{\partial n^2}\Big|_{n=0} + \frac{1}{\lambda} \sum_{p=1}^{\infty} M_p(\zeta) c_p(s) + \lambda \sum_{k=1}^{\infty} \frac{\sin \theta_k \zeta}{\theta_k} \Big(\frac{\partial^2 B_k}{\partial n \partial s}\Big|_{n=0} - \frac{R}{\rho} \frac{\partial b_k(s)}{\partial s} \Big) \tag{7}$$

$$T = \sum_{i=1}^{2} \sum_{m=0}^{\infty} A_m^{(i)} J_0^{-2} \Lambda_i \Big[\frac{\partial^2}{\partial \xi \partial \eta} - \frac{1}{\operatorname{ch} 2\xi_0 - \cos 2\eta} \Big(\sin 2\eta \frac{\partial}{\partial \xi} + \operatorname{sh} 2\xi_0 \frac{\partial}{\partial \eta} \Big) \Big] \operatorname{Ce}_{2m}(q_i, \xi)\big|_{\xi=\xi_0} \operatorname{ce}_{2m}(q_i, \eta) +$$
$$\lambda \sum_{p=1}^{\infty} Q_p(\zeta) \Big(\frac{\partial^2 C_p}{\partial n \partial s}\Big|_{n=0} - \frac{R}{\rho} \frac{\partial c_p}{\partial s} \Big) + \frac{\lambda}{2} \sum_{k=1}^{\infty} \frac{\sin \theta_k \zeta}{\theta_k} \Big(\frac{\partial^2 b_k}{\partial s^2} + \frac{R}{\rho} \frac{\partial B_k}{\partial n}\Big|_{n=0} - \frac{\partial^2 B_k}{\partial n^2}\Big|_{n=0} \Big)$$

$$Z = \sum_{i=1}^{2} \sum_{m=0}^{\infty} A_m^{(i)} (\mu_i + \alpha_i^2)(\cos \lambda \alpha_i \zeta \cos \lambda \beta_i - \cos \lambda \beta_i \zeta \cos \lambda \alpha_i) J_0^{-2} \times$$
$$\frac{\partial}{\partial \xi} \operatorname{Ce}_{2m}(q_i, \xi)\big|_{\xi=\xi_0} \operatorname{ce}_{2m}(q_i, \eta) + \sum_{p=1}^{\infty} F_p(\zeta) \frac{\partial C_p}{\partial n}\Big|_{n=0} + \frac{1}{2} \sum_{k=1}^{\infty} \cos \theta_k \zeta \frac{\partial b_k}{\partial s}$$

$$Q_p(\zeta) = \lambda^{-2}\left[2\varkappa_p \sin\varkappa_p\zeta\cos\delta_p - (\gamma_p{}^2 + \varkappa_p{}^2)\frac{\sin\delta_p\zeta}{\delta_p}\cos\varkappa_p\right]$$

$$M_p(\zeta) = \frac{\Omega^2\nu}{2(1-\nu)}(\gamma_p{}^2 + \varkappa_p{}^2)\frac{\sin\delta_p\zeta}{\delta_p}\cos\varkappa_p$$

$$F_p(\zeta) = \lambda^{-2}(\gamma_p{}^2 + \varkappa_p{}^2)(\cos\varkappa_p\zeta\cos\delta_p - \cos\delta_p\zeta\cos\varkappa_p)$$

$$J_0^2 = \frac{\varepsilon^2}{2(1-\varepsilon^2)}(\mathrm{ch}\,2\xi_0 - \cos 2\eta),$$

$$s = \frac{1}{2(1-\varepsilon^2)}\int\limits_0^\eta [2-\varepsilon^2-\varepsilon^2\cos 2\eta]^{1/2} d\eta$$

$$\theta_k{}^2 = \sigma_k{}^2 + \lambda^2\Omega^2, \quad \varkappa_p{}^2 = \gamma_p{}^2 + \lambda^2\Omega^2, \quad \delta_p{}^2 = \gamma_p{}^2 + \frac{1-2\nu}{2(1-\nu)}\lambda^2\Omega^2$$

It can easily be shown that the functions $M_p(\zeta)$, $Q_p(\zeta)$, $F_p(\zeta)$ are of zero order with respect to λ. Thus the problem in question reduces to that of determining the constants $A_m^{(i)}$ and the boundary values of the functions C_p and B_k, from the stresses given at the side surface.

The infinite system of equations for determining $A_m^{(i)}$, $c_p(s)$ and $b_k(s)$ can be constructed by expanding the stresses given in (7) into the Fourier series in ξ. We have

$$\sum_{i=1}^{2}\sum_{m=0}^{\infty} A_m^{(i)}\Big\{R_i^{(r)}J_0^{-2}\Big[a_{2m} + \mu_i\frac{2-\varepsilon^2}{2(1-\varepsilon^2)} - \tag{8}$$
$$\frac{1}{\mathrm{ch}\,2\xi_0 - \cos 2\eta}\Big(\mathrm{sh}2\xi_0\frac{\partial}{\partial\xi} - \sin 2\eta\frac{\partial}{\partial\eta}\Big)\Big] + P_i^{(r)}\Big\}\mathrm{Ce}_{2m}(q_i,\xi)\big|_{\xi=\xi_0}\times$$
$$\mathrm{ce}_{2m}(q_i,\eta) + \lambda\sum_{p=1}^{\infty} G_p^{(r)}\frac{\partial^2 C_p}{\partial n^2}\Big|_{n=0} + \frac{1}{\lambda}\sum_{p=1}^{\infty} L_p^{(r)}c_p(s) +$$
$$\lambda\sum_{k=1}^{\infty} U_k^{(r)}\Big(\frac{\partial^2 B_k}{\partial n\partial s}\Big|_{n=0} - \frac{R}{\rho}\frac{\partial b_k(s)}{\partial s}\Big) = N_r$$

$$\sum_{i=1}^{2}\sum_{m=0}^{\infty} A_m^{(i)}R_i^{(r)}J_0^{-2}\Big[\frac{\partial^2}{\partial\xi\partial\eta} - \frac{1}{\mathrm{ch}\,2\xi_0 - \cos 2\eta}\Big(\sin 2\eta\frac{\partial}{\partial\xi} +$$
$$\mathrm{sh}\,2\xi_0\frac{\partial}{\partial\eta}\Big)\Big]\mathrm{Ce}_{2m}(q_i,\xi)\big|_{\xi=\xi_0}\mathrm{ce}_{2m}(q_i,\eta) +$$
$$\lambda\sum_{p=1}^{\infty} C_p^{(r)}\Big(\frac{\partial^2 C_p}{\partial n\partial s}\Big|_{n=0} - \frac{R}{\rho}\frac{\partial c_p}{\partial s}\Big) +$$
$$\frac{\lambda}{2}\sum_{k=1}^{\infty} U_k^{(r)}\Big(\frac{\partial^2 b_k}{\partial s^2} + \frac{R}{\rho}\frac{\partial B_k}{\partial n}\Big|_{n=0} - \frac{\partial^2 B_k}{\partial n^2}\Big|_{n=0}\Big) = T_r$$

$$\lambda\sum_{i=1}^{2}\sum_{m=0}^{\infty} A_m^{(i)}V_i^{(r)}\frac{\partial}{\partial\xi}\mathrm{Ce}_{2m}(q_i,\xi)\big|_{\xi=\xi_0}\mathrm{ce}_{2m}(q_i,\eta) +$$
$$\sum_{p=1}^{\infty} K_p^{(r)}\frac{\partial C_p}{\partial n}\Big|_{n=0} + \sum_{k=1}^{\infty} W_k^{(r)}\frac{\partial b_k}{\partial s} = Z_r \quad (r = 1, 2, \ldots)$$

$$R_i^{(r)} = 4(-1)^r \pi r \Omega^2 \frac{[\lambda^2(\sigma-1)\mu_i + \lambda^2\sigma\alpha_i^2 - r^2\pi^2]\,\alpha_i \sin\lambda\alpha_i \cos\lambda\beta_i}{(\lambda^2\alpha_i^2 - r^2\pi^2)(\lambda^2\beta_i^2 - r^2\pi^2)(\mu_i + \alpha_i^2)}$$

$$P_i^{(r)} = (-1)^r \pi r \Omega^2 (1-2\sigma)\frac{(\mu_i + \alpha_i^2)\cos\lambda\alpha_i \sin\lambda\beta_i}{\beta_i(\lambda^2\beta_i^2 - r^2\pi^2)}$$

$$V_i^{(r)} = 2(-1)^r \Omega^2 \frac{\{\lambda^2\beta_i^2\Omega^2 + (r-1)^2\pi^2[4\mu_i(\sigma-1) - \Omega^2]\}\,\alpha_i \sin\lambda\alpha_i \cos\lambda\beta_i}{[\lambda^2\alpha_i^2 - (r-1)^2\pi^2][\lambda^2\beta_i^2 - (r-1)^2\pi^2](\mu_i + \alpha_i^2)}$$

$$G_p^{(r)} = 4(-1)^r r\pi\Omega^2 \frac{[(\sigma-1)\gamma_p^2 + \sigma\varkappa_p^2 - r^2\pi^2]\,\varkappa_p \sin\varkappa_p \cos\delta_p}{(\varkappa_p^2 - r^2\pi^2)(\delta_p^2 - r^2\pi^2)(\gamma_p^2 + \varkappa_p^2)}$$

$$L_p^{(r)} = (-1)^r r\pi\Omega^2(1-2\sigma)\frac{(\gamma_p^2 + \varkappa_p^2)\sin\delta_p \cos\varkappa_p}{\delta_p(\delta_p^2 - r^2\pi^2)}$$

$$K_p^{(r)} = 2(-1)^r \Omega^2 \frac{\{\delta_p^2\lambda^2\Omega^2 + (r-1)^2\pi^2[4\gamma_p^2(\sigma-1) - \lambda^2\Omega^2]\}\,\varkappa_p \sin\varkappa_p \cos\delta_p}{[\varkappa_p^2 - (r-1)^2\pi^2][\delta_p^2 - (r-1)^2\pi^2](\gamma_p^2 + \varkappa_p^2)}$$

$$U_k^{(r)} = \frac{2(-1)^{r+k+1} r\pi}{\theta_k(\theta_k^2 - r^2\pi^2)}, \quad W_k^{(r)} = \frac{(-1)^{r+k+1}\theta_k}{\theta_k^2 - (r-1)^2\pi^2}, \quad \sigma = \frac{1-2\nu}{2(1-\nu)}$$

Here a_{2m} are the eigenvalues of the Mathieu functions, while $N_r(s)$, $T_r(s)$ and $Z_r(s)$ are coefficients of the Fourier expansions of the boundary functions $N(\zeta, s)$, $T(\zeta, s)$, $Z(\zeta, s)$.

Let us assume that all functions $X(s) = N_r(s)$, $T_r(s)$, $Z_r(s)$, $c_p(s)$, $b_k(s)$ can be written in the form of the power series in λ

$$X(s) = X_0(s) + \lambda X_1(s) + \lambda^2 X_2(s) + \ldots \tag{9}$$

According to [3] we have

$$\left.\frac{\partial C_p}{\partial n}\right|_{n=0} = \frac{1}{\lambda}\left\{\gamma_p c_{p0} + \lambda\left(\gamma_p c_{p1} - \frac{R}{2\rho}c_{p0}\right) + \lambda^2\left[\gamma_p c_{p2} - \frac{R}{2\rho}c_p - \left(\frac{R^2}{8\gamma_p\rho^2}c_{p0} + \frac{1}{2\gamma_p}\frac{\partial^2 c_{p0}}{\partial s^2}\right)\right] + \ldots\right\} \tag{10}$$

$$\left.\frac{\partial^2 C_p}{\partial n^2}\right|_{n=0} = \frac{1}{\lambda^2}\left\{\gamma_p^2 c_{p0} + \lambda\left(\gamma_p^2 c_{p1} - \frac{\gamma_p R}{\rho}c_{p0}\right) + \lambda^2\left[\gamma_p^2 c_{p2} - \frac{\gamma_p R}{\rho}c_{p1} + \left(\frac{R^2}{2\rho^2}c_{p0} - \frac{\partial^2 c_{p0}}{\partial s^2}\right)\right] + \ldots\right\}$$

We obtain the expressions for the normal derivatives of the functions $B_k(n, s)$ on the contour Γ by replacing in (10) γ_p by σ_k and $c_{pi}(s)$ by $b_{ki}(s)$. Substituting the asymptotic expansions (9) and (10) into (8) and equating the coefficients of like powers in λ we obtain, in the first stage, the following infinite system of linear equations for determining $c_{p0}(s)$ and $b_{k0}(s)$:

$$\sum_{p=1}^{\infty}[G_p^{(r)}\gamma_p^2 + L_p^{(r)}]\,c_{p0}(s) = 0, \quad -\frac{1}{2}\sum_{k=1}^{\infty}U_k^{(r)}\sigma_k^2 b_{k0}(s) = 0 \tag{11}$$

$$\sum_{p=1}^{\infty}K_p^{(r)}\gamma_p c_{p0}(s) = 0 \quad (r = 1, 2, \ldots)$$

Inspecting the system (11) we find, that it has only a zero solutions $c_{p0}(s) \equiv 0$, $b_{k0}(s) \equiv 0$.

In the second stage of the asymptotic process we obtain a system for determining $A_m^{(i)}$, $c_{p1}(s)$, $b_{k1}(s)$

$$\sum_{i=1}^{2}\sum_{m=0}^{\infty} A_m^{(i)}\Big\{R_i^{(r)}J_0^{-2}\Big[a_{2m}+\mu_i\frac{2-\varepsilon^2}{2(1-\varepsilon^2)}-\frac{1}{\operatorname{ch}2\xi_0-\cos 2\eta}\times \tag{12}$$

$$\Big(\operatorname{sh}2\xi_0\frac{\partial}{\partial\xi}-\sin 2\eta\frac{\partial}{\partial\eta}\Big)+P_i^{(r)}\Big\}\Big]\operatorname{Ce}_{2m}(q_i,\xi)\big|_{\xi=\xi_0}\operatorname{ce}_{2m}(q_i,\eta)+$$

$$\sum_{p=1}^{\infty}(G_p^{(r)}\gamma_p^2+L_p^{(r)})\,c_{p1}(s)=N_{r0}$$

$$\sum_{p=1}^{\infty}K_p^{(r)}\gamma_p c_{p1}(s)=Z_{r0}$$

$$-\frac{1}{2}\sum_{k=1}^{\infty}U_r^{(k)}\sigma_k^2 b_{k1}(s)=$$

$$T_{r0}-\sum_{i=1}^{2}\sum_{m=0}^{\infty}A_m^{(i)}R_i^{(r)}\Big[\frac{\partial^2}{\partial\xi\partial\eta}-\frac{1}{\operatorname{ch}2\xi_0-\cos 2\eta}\Big(\sin 2\eta\frac{\partial}{\partial\xi}+$$

$$\operatorname{sh}2\xi_0\frac{\partial}{\partial\eta}\Big)\Big]\operatorname{Ce}_{2m}(q_i,\xi)\big|_{\xi=\xi_0}\operatorname{ce}_{2m}(q_i,\eta)\quad(r=1,2,\ldots)$$

In the third stage we find $c_{p2}(s)$ and $b_{k2}(s)$

$$\sum_{p=1}^{\infty}(G_p^{(r)}\gamma_p^2+L_p^{(r)})\,c_{p2}=N_{r1}+\sum_{p=1}^{\infty}G_p^{(r)}\frac{R\gamma_p}{\rho}c_{p1}-\sum_{k=1}^{\infty}U_k^{(r)}\sigma_k b_{k1} \tag{13}$$

$$\sum_{p=1}^{\infty}K_p^{(r)}\gamma_p c_{p2}=Z_{r1}-\sum_{i=1}^{2}\sum_{m=0}^{\infty}A_m^{(i)}V_i^{(r)}\frac{\partial}{\partial\xi}\operatorname{Ce}_{2m}(q_i,\xi)\big|_{\xi=\xi_0}\operatorname{ce}_{2m}(q_i,\eta)+$$

$$\sum_{p=1}^{\infty}K_p^{(r)}\frac{R}{2\rho}c_{p1}-\sum_{k=1}^{\infty}W_k^{(r)}b_{k1}'$$

$$-\frac{1}{2}\sum_{k=1}^{\infty}U_k^{(r)}\sigma_k^2 b_{k2}=T_{r1}-\sum_{p=1}^{\infty}G_p^{(r)}\gamma_p c_{p1}-$$

$$\sum_{k=1}^{\infty}U_k^{(r)}\frac{R}{\rho}\sigma_k b_{k1}\quad(r=1,2,\ldots)$$

etc. When the loading is sufficiently smooth, the process can be continued for as long as we please.

At this stage we can carry out the asymptotic analysis of the stress-strain state of the plate inside the region, and at the boundary. We have

$$\sigma_s=2G\Big\{\sigma_s^{(0)}+\sum_{p=1}^{\infty}M_p(\zeta)\,H^{-1/2}c_{p1}(s)\exp\frac{\gamma_p n}{\lambda}+ \tag{14}$$

$$\lambda\Big[\sum_{p=1}^{\infty}M_p(\zeta)\,(H^{-1/2}c_{p2}(s)+Sc_{p1}(s))\exp\frac{\gamma_p n}{\lambda}+$$

$$\sum_{p=1}^{\infty} Q_p(\zeta) H^{-3/2} \frac{R}{\rho} \gamma_p c_{p1}(s) \exp\frac{\gamma_p n}{\lambda} - \sum_{k=1}^{\infty} \frac{\sin\theta_k\zeta}{\theta_k} H^{-3/2} \times$$

$$\sigma_k b_{k1}(s) \exp\frac{\sigma_k n}{\lambda}\Big] + \ldots\Big\} e^{i\omega t}$$

$$\tau_{nz} = 2G\Big\{\sum_{p=1}^{\infty} F_p(\zeta) H^{-1/2}\gamma_p c_{p1}(s) \exp\frac{\gamma_p n}{\lambda} + \lambda\Big[\tau_{nz}^{(0)} + \sum_{p=1}^{\infty} F_p(\zeta) \times$$

$$\Big(H^{-1/2}\gamma_p c_{p2}(s) + \gamma_p S c_{p1}(s) - H^{-3/2}\frac{R}{2\rho} c_{p1}(s)\Big) \exp\frac{\gamma_p n}{\lambda} +$$

$$\frac{1}{2}\sum_{k=1}^{\infty} H^{-3/2}\cos\theta_k\zeta b_{k1}(s) \exp\frac{\sigma_k n}{\lambda}\Big] + \ldots\Big\} e^{i\omega t}$$

$$W = \Big\{w^{(0)} + R\lambda\sum_{p=1}^{\infty} D_p(\zeta) H^{-1/2} c_{p1}(s) \exp\frac{\gamma_p n}{\lambda} + \ldots\Big\} e^{i\omega t}$$

$$S = \frac{1}{\sqrt{H}}\Big\{-\frac{1}{2\gamma_p}\Big(\frac{l^2}{4} + \frac{\partial^2}{\partial s^2}\Big)\frac{n}{H} + \frac{1}{2\gamma_p}\Big(\frac{l''}{4} + l'\frac{\partial}{\partial s}\Big)\frac{n^2}{H^2} -$$

$$\frac{5l'^2}{24\gamma_p}\frac{n^3}{H^3}\Big\}, \quad l = \frac{R}{\rho}$$

$$D_p(\zeta) = [2\gamma_p^2 \cos\varkappa_p\zeta \cos\delta_p - (\gamma_p^2 + \varkappa_p^2)\cos\delta_p\zeta \cos\varkappa_p]\lambda^{-2}$$

The formulas $w^{(0)}$, $\tau_{nz}^{(0)}$, $\sigma_s^{(0)}$ in (14) correspond to the penetrating solution expressed in terms of the Mathieu functions. This implies that the correction brought into the determination of W by the boundary layers, is of higher order in λ than W itself over the whole region occupied by the plate. The same can be said of the displacements U_n and U_s. The stresses present a somewhat different picture. When the stress σ_s (σ_n, τ_{ns}) is considered away from the edge, then the influence of the boundary layers can be neglected. As the side surface of the plate $(n = 0)$ the correction due to the boundary layer is,, generally, of the same order in λ as the penetrating solution. In the case of the stress τ_{nz} (τ_{sz}) the boundary layers play a fundamental part at the boundary.

The infinite system (12) can be truncated to a finite system containing $2L$ equations. Every equation is written at a finite number of points of the contour Γ, and L is the number of the coefficients in the Fourier expansions of the boundary functions. The number of the coeffcients $A_m^{(i)}$ to be determined coincides with the number of points of partition of the arc s $(0 \leqslant \eta \leqslant \pi / 2)$. The practical convergence of the process of determining the coefficients $A_m^{(i)}$ and of the series containing $A_m^{(i)}$ was studied, with the number of the boundary layers retained in (12) and (13) equal to $L - 1$. The number L was chosen so as to satisfy the boundary conditions at the side of the plate with an error not exceeding 1%.

To illustrate the scope of the method proposed we investigate the case of the forced steady-state oscillations of an elliptic plate for the following values of the boundary functions:

$$N = \alpha\zeta, \quad T = 0, \quad Z = 0 \qquad (15)$$

In the case of deformations, skew symmetric with respect to the middle surface of the plate, the amplitude function of transverse displacement has the form

$$w = R \sum_{i=1}^{2} \sum_{m=0}^{\infty} A_m^{(i)} [2\mu_i \cos \lambda\alpha_i\zeta \cos \lambda\beta_i - (\mu_i + \alpha_i^2) \cos \lambda\beta_i\zeta \cos \lambda\alpha_i] \times$$
$$\mathrm{Ce}_{2m}(q_i, \xi)\, \mathrm{ce}_{2m}(q_i, \eta) + \lambda R \sum_{p=1}^{\infty} D_p(\zeta) H^{-1/2} c_{p1}(s) \exp \frac{\gamma_p n}{\lambda} + \ldots$$

Computations were carried out at $\nu = 1/3$ using a digital computer. The extent to which the boundary conditions were satisfied at the side surface of the plate in the first stage of the asymptotic process was verified for various numbers of the boundary layers, while varying the parameters λ, Ω and ε.

Solid lines in Fig. 2 depict the amplitude stress function σ_n ($\sigma_n{}^* = (2G\alpha)^{-1}\sigma_n e^{-i\omega t}$) for $\lambda = 0.3$, $\Omega = 2.2$ and $\varepsilon = 0.3$. Line *1* depicts the penetrating solution, line *2* a solution including the first boundary layer, and line *3* a solution with three boundary layers, the latter practically coinciding with N. The numerical analysis has shown that the first boundary layer brings in a significant contribution, and taking into account the remaining boundary layers makes it possible to satisfy the boundary conditions at the side surface of the plate with a predetermined accuracy.

In order to analyse the convergence of the asymptotic series in λ in practice, we compared wthe amplitude stress function σ_s at the boundary of an elliptic plate of small eccentricity ($\varepsilon = 0.01$) with the corresponding stress for a circular plate expressed in the terms of series in Bessel functions [2, 6]. Broken lines in Fig. 2 depict the amplitude stress function σ_s ($\sigma_s{}^* = (2G\alpha)^{-1}\sigma_s e^{i\omega t}$) at the boundary for $\lambda = 0.3$, $\Omega = 2.2$ and $\varepsilon = 0.01$ computed at the first stage of the asymptotic process. Numbers *1* and *2* denote, respectively, the penetrating solution and the solution with three boundary layers, the latter practically coinciding with the corresponding stress for a circular plate.

The proposed method enables us to compute the eigenfrequencies of the elliptic plate. What we do, is to consider a collection of problems with the following boundary conditions:

$$\begin{aligned} &N = \mathrm{ce}_m(\eta), \quad T = 0, \quad Z = 0 \\ &N = \mathrm{se}_{m+1}(\eta), \quad T = 0, \quad Z = 0 \\ &N = 0, \quad T = \mathrm{ce}_m(\eta), \quad Z = 0 \\ &N = 0, \quad T = \mathrm{se}_{m+1}(\eta), \quad Z = 0, \qquad m = 0, 1, 2 \ldots \end{aligned} \tag{16}$$

etc. Obviously, superimposing the solutions of these problems we can obtain a solution of a problem with arbitrary boundary conditions at the side surface of the plate [7]. The eigenfrequencies are determined in the given interval of variation of Ω for every separate problem (16) is the same manner as that for the boundary conditions (15).

The eigenvalues are determined as those values of the forced oscillation frequency, for which the boundary value problem in question has no finite solution. The actual determination of the intervals for the eigenfrequencies of the elliptic plate is based on the analysis of the dependence of the function $w(0, y, 0)$ on the frequency Ω. Fig. 3 shows the graphs of this function for $\lambda = 0.3$, $\varepsilon = 0.3$ with the numbers *1*—*5* corresponding to the frequencies 1.8, 1.9, 2.0, 4.8 and 4.9. The graphs make it possible to determine the intervals of the eigenfrequencies $\Omega_1 \in (1.8;\ 1.9)$, $\Omega_2 \in (4.8;\ 4.9)$, and the corresponding eigenfrequencies for the circular plates are

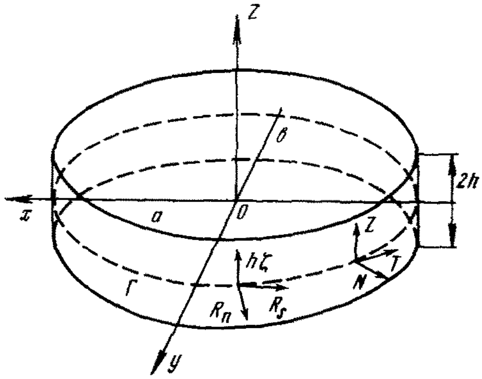

Fig. 1

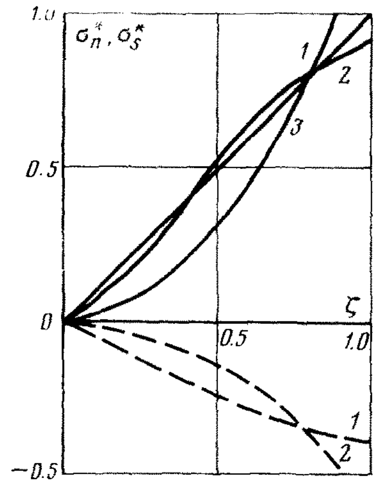

Fig. 2

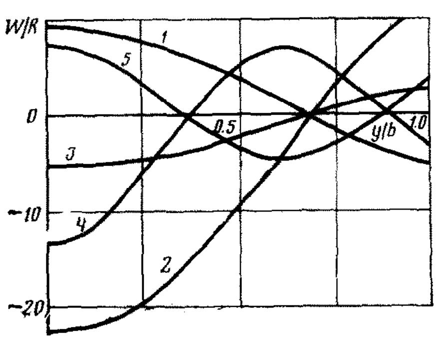

Fig. 3

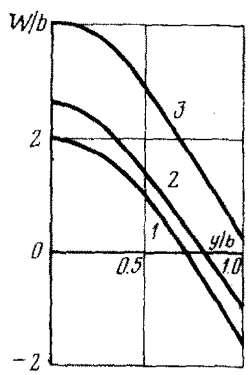

Fig. 4

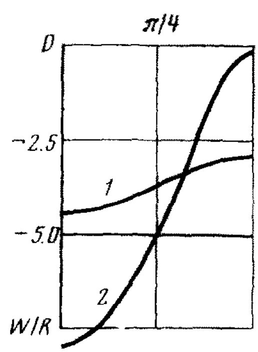

Fig. 5

$(\lambda = 0.3)$ $\Omega_1 = 1.95$, $\Omega_2 = 4.96$.

We have analysed the behavior of the functions $w(0, y, 0)$ and $w(x, 0, 0)$ when the circle was deformed into an ellipse, inscribed or circumscribed. Numerical analysis has shown that for small values of the eccentricity (from 0.01 to 0.1) the displacements w of the elliptic plate practically coincide with the transverse displacement of the circular plate computed using the method given in [2, 6]. Fig. 4 depicts the graphs of the functions $w(0, y, 0)$ for $h/b = 0.1$, $\Omega = 0.5$ and $\varepsilon = 0.01$, 0.5 and 0.6 (curves *1*, *2* and *3* respectively).

The behavior of $w(x, y, 0)$ relative to the angle η at the boundary Γ of the plate was also studied. For small ε $(\varepsilon \leqslant 0.1)$ the displacements of the plate contour can be assumed constant. The dependence of w on η increased with increasing ε. Fig. 5 shows the graph of $w(x, y, 0)$ versus η for $n = 0$, for the values $\lambda = 0.05$, $\Omega = 0.2$, for $\varepsilon = 0.4$ and 0.5 (curves *1* and *2* respectively).

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